Spinors on a curved noncommutative space: coupling to torsion and the Gross-Neveu model

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Abstract

We analyse the spinor action on a curved noncommutative space, the so-called truncated Heisenberg algebra, and in particular, the nonminimal coupling of spinors to the torsion. We find that dimensional reduction of the Dirac action gives the noncommutative extension of the Gross-Neveu model, the model which is, as shown by Vignes-Tourneret, fully renormalisable.

1 Introduction

A noncommutative model which attracted much attention and initiated a great amount of work in the past decade is the Grosse-Wulkenhaar (GW) model [1, 2]. It describes a real scalar field ϕ on the noncommutative Moyal space evolving in the external oscillator potential, in two and four euclidean dimensions,*

$$\mathcal{S}_{GW} = \int \frac{1}{2}\,(\partial_\mu\phi)(\partial^\mu\phi) + \frac{1}{2}\,\Omega^2\tilde{x}_\mu\tilde{x}^\mu\phi^2 + \frac{m^2}{2}\,\phi^2 + \frac{\lambda}{4}\,\phi^4. \tag{1.1}$$

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*As we do not specify the representation here, we have only one product: the one which defines the algebra, (2.6).

The model has a number of exceptionally good properties in quantisation which have been established and analysed in many papers since 2003, and include perturbative renormalisability to all orders and vanishing of the β-function at the self-duality point [3, 4]; the model is likely to be perturbatively solvable. There is a considerable progress in nonperturbative treatment as well; for recent results and developments, see [5, 6, 7]. Athough it was initially treated in the matrix base, the Grosse-Wulkenhaar model was subsequently analysed by the multiscale analysis in the coordinate base [8, 9, 10], and that analysis revealed many interesting mathematical properties and enabled generalisations. Similar, though in many aspects different models of fields in the external magnetic potential were proposed even before [11, 12] as exactly solvable quantum field theories; one of the most important properties which these models possess is the Langmann-Szabo (LS) duality, a new kind of symmetry which is present also in the GW model.

Many attempts have been made to understand the physical reasons underlying renormalisability of the GW model and to generalise it to other physical fields, in particular to spinor and gauge fields. One way of generalisation is straightforward: by constructing Lagrangians which have the Mehler kernel as propagator. In the case of spinors this was done successfully in [13]: the proposed spinor action is

$$\mathcal{S}_8 = \int \bar\psi \mathcal{D}_8 \psi = \int \bar\psi \big(i\Gamma^\mu \partial_\mu + \Omega \Gamma^{\mu+4} \tilde{x}_\mu \big) \psi . \tag{1.2}$$

This action is defined on the space of spinors $\psi(x^\mu)$, $\mu = 1, \dots 4$, which carry a double-dimensional spinor representation: $\{\Gamma^k, \Gamma^l\} = 2\delta^{kl}$, $k, l = 1, \dots 8$. The square of the Dirac operator $\mathcal{D}_8$ gives, up to a constant coordinate-independent matrix Σ, exactly the Hamiltonian of the massless GW model. Consequently, the spinor action (1.2) is renormalisable. The other possibility of 'taking the square root' of the harmonic potential was proposed in [14, 15]. The 2d action which was discussed,

$$\mathcal{S}_{nGN} = \int \bar\psi \big(-i\gamma^\mu \partial_\mu + \Omega \gamma^\mu \tilde{x}_\mu + \tilde{m} + \kappa \gamma_5 \big) \psi - \frac{g_A}{4} \mathcal{J}_A^2 , \tag{1.3}$$

is a noncommutative extension of the Gross-Neveu (GN) model, [16]; the $\mathcal{J}_A$ are the currents bilinear in the fermionic field. Remarkably, this action is renormalisable too; the parity breaking γ_5-term appears as counterterm when the fermions are massive.

Generalisation of the GW model to gauge fields has been more difficult, and indeed a construction of a renormalisable gauge model is still an unsolved problem. At the first sight the problem is easy to understand. In order to have an oscillator-type external potential, and correspondingly the Mehler kernel propagator, one has to include coordinate-dependent terms into the action: but coordinate-dependent terms break the gauge symmetry. This particular problem however in noncommutative geometry can be solved surprisingly simply. Namely, the momentum operators p_μ which define the differential can and often do belong to the algebra of coordinates $\mathcal{A}$. For example on a space with constant nondegenerate noncommutativity

$$[x^\mu, x^\nu] = i\hbar J^{\mu\nu} = \text{const}, \tag{1.4}$$

(of which the Moyal space is a representation), the momenta are given by

$$p_\mu = (i\bar{k}J)^{-1}_{\mu\nu}\, x^\nu . \tag{1.5}$$

Then covariant derivatives, or more precisely covariant momenta $\tilde{\mathsf{X}}_\mu = p_\mu + A_\mu$, also belong to the space $\mathcal{A}$ and moreover transform covariantly, in the adjoint representation of the gauge group. (Here by A_μ we denote the potential which corresponds to the gauge group: it is in the literature usually the noncommutative $U(1)$ or $U(N)$.) Thus using $\tilde{\mathsf{X}}_\mu$ or covariant coordinates, $\mathsf{X}^\mu = x^\mu + i\bar{k}J^{\mu\nu}A_\nu$, one can define gauge invariant actions [17, 18, 19, 20, 21]. Still, additional physical tools to construct the prescribed generalisation of the Yang-Mills action are needed. In [22], the oscillator potential was introduced through the ghost sector. A very promising action was obtained in [23, 24] as an effective action for the $U(1)$ gauge field coupled to the GW scalar, after integration of the scalar modes. Though these models can be written in terms of the covariant coordinates and possess the LS duality, they have difficulties related mainly to the vacuum structure and none has proved to be renormalisable, [25, 26, 27, 28, 29]. Comprehensive recent reviews of the gauge models are for example, [29, 30].

Another logic of generalisation of the Grosse-Wulkenhaar model was proposed in [31]: it is based on the observation that the harmonic potential can be seen as the scalar curvature of an appropriately defined noncommutative space. This geometric interpretation gives a straightforward way to obtain the action for various fields: it is simply the action on a curved spacetime. There are however additional details. Since two-dimensional space (1.4) can be considered as a contraction of a three-dimensional algebra (the 'truncated Heisenberg algebra') which has finite-dimensional matrix representations, the corresponding action can be understood as geometric, or geometrically consistent regularisation. On the level of geometry, we use a kind of Kaluza-Klein (KK) reduction followed by rescaling or renormalisation of the physical fields. Apart from reinterpretation of the GW model [31, 32], this approach gave an interesting gauge model [33], with an improved vacuum structure. The model is however relatively complicated as due to the KK reduction it contains interacting gauge and scalar fields: a report on the present status of the calculations will be published elsewhere. Attempts to define the fermion action in the geometric framework were initially not successfull in the sense that coordinate-dependent terms were absent. As we shall see the reason was simple: we treated only fermions minimally coupled to gravity. But even in commutative case the minimal coupling, applied in two dimensions, removes the explicit dependence on the connection, [34]. The solution to this problem is to couple fermions to the torsion nonminimally: a construction of the corresponding model is the main content of this paper, and as we shall see, it gives as result exactly the noncommutative generalisation of the Gross-Neveu model studied in [14].

The plan of the paper is the following: in Section 2 we review briefly properties of the truncated Heisenberg space necessary for our construction and calculate the torsion. In Section 3 we introduce the action for the massive Dirac fermions and the nonminimal coupling terms and reduce these actions to two dimensions. In Section 4 we discuss our results.

2 The truncated Heisenberg space

We will shortly introduce the main geometric objects which are of relevance here. The truncated Heisenberg algebra is defined by commutation relations

$$\begin{aligned}
&[\mu x, \mu y] = i\epsilon(1-\mu z)\\
&[\mu x, \mu z] = i\epsilon(\mu y\,\mu z + \mu z\,\mu y) \qquad (2.6)\\
&[\mu y, \mu z] = -i\epsilon(\mu x\,\mu z + \mu z\,\mu x).
\end{aligned}$$

The μ is a constant of dimension of the inverse length; ϵ is a dimensionless parameter which indicates the strength of noncommutativity; we denote $\bar{k} = \epsilon\mu^{-2}$. For $\epsilon = 1$ algebra (2.6) can be represented by $n\times n$ matrices for any integer n, [31]; $\epsilon = 0$ is the commutative 'limit'. We usually assume that parameters can be taken as independent: μ, related to some relevant length or mass scale (like for example the cosmological constant in the gravitational case), and ϵ related purely to noncommutativity; one can alternatively assume just that $\bar{k} = l_{Pl}^2$.

Contraction $\mu\to 0$ gives the Heisenberg algebra,

$$[x^\mu, x^\nu] = i\bar{k}\epsilon^{\mu\nu}, \qquad \mu,\nu = 1,2, \qquad (2.7)$$

which has only infinite-dimensional representations. The relation between the Heisenberg algebra and the truncated Heisenberg algebra can be seen in the Fock space representation of the former as truncation of infinite matrices to the finite ones: it is a weak limit. In many aspects however it is consistent to treat this limit as a reduction to the subspace $z = 0$ of the initial noncommutative space, [35].

Symmetries of algebra (2.6) are rotations in the xy-plane: the generator is $M = \mu^2x^2+\mu^2y^2+\mu z$.† Parity on the other hand is not a symmetry of the truncated Heisenberg algebra, and parity breaking we shall see remains in the spinor Lagrangian.

Apart from coordinates one can define derivations and p-forms. In the approach which we are using the space of 1-forms is spanned by frame $\{\theta^\alpha\}$, [20]

$$[f, \theta^\alpha] = 0. \qquad (2.8)$$

Dual to θ^α are derivations e_α, $\theta^\alpha(e_\beta) = \delta^\alpha_\beta$. The differential d of function f can be defined as

$$df = (e_\alpha f)\,\theta^\alpha. \qquad (2.9)$$

Derivations e_α are inner in the finite-matrix spaces, generated by elements $p_\alpha \in \mathcal{A}$

$$e_\alpha f = [p_\alpha, f] \qquad (2.10)$$

which we call the momenta. We shall assume that e_α are always of the form (2.10) and in addition that p_α are antihermitian. Condition (2.8) enables in fact to introduce

†On $z = 0$ this generator, interestingly, reduces to $M| = \mu^2x^2+\mu^2y^2 = i\epsilon(xp_2 - yp_1)$.

consistently the metric which has, in the frame basis, constant components. In our particular geometry this metric is euclidean,

$$g_{\alpha\beta} = \delta_{\alpha\beta}, \qquad \alpha, \beta = 1, 2, 3. \tag{2.11}$$

As seen from (2.9) the choice of momenta is equivalent to the choice of d: therefore the differential calculus is neither uniquely fixed, nor do we have a canonical choice like the de Rham calculus in commutative geometry. We choose for the truncated Heisenberg space, [31]

$$\epsilon p_1 = i\mu^2 y, \qquad \epsilon p_2 = -i\mu^2 x, \qquad \epsilon p_3 = i\mu(\mu z - \frac{1}{2}), \tag{2.12}$$

so that on $z = 0$ the differential reduces to the standard differential of the Heisenberg space.

The momentum algebra can be used to define the exterior product of 1-forms, and to extend this product to 2-forms, 3-forms and so on, [20]. In the truncated Heisenberg geometry we obtain the following relations

$$\begin{aligned} &(\theta^1)^2 = 0, \qquad (\theta^2)^2 = 0, \qquad (\theta^3)^2 = 0, \qquad \{\theta^1, \theta^2\} = 0, \\ &\{\theta^1, \theta^3\} = i\epsilon(\theta^2\theta^3 - \theta^3\theta^2), \qquad \{\theta^2, \theta^3\} = i\epsilon(\theta^3\theta^1 - \theta^1\theta^3). \end{aligned} \tag{2.13}$$

From (2.13) and associativity of the exterior product follow the rules of multiplication of three 1-forms:

$$\begin{aligned} &\theta^1\theta^3\theta^1 = \theta^2\theta^3\theta^2, \qquad \theta^3\theta^1\theta^3 = 0, \qquad \theta^3\theta^2\theta^3 = 0, \\ &\theta^1\theta^2\theta^3 = -\theta^2\theta^1\theta^3 = \theta^3\theta^1\theta^2 = -\theta^3\theta^2\theta^1 = i\,\frac{\epsilon^2 - 1}{2\epsilon}\,\theta^2\theta^3\theta^2, \\ &\theta^1\theta^3\theta^2 = -\theta^2\theta^3\theta^1 = i\,\frac{\epsilon^2 + 1}{2\epsilon}\,\theta^2\theta^3\theta^2. \end{aligned} \tag{2.14}$$

Obviously, there is only one (linearly independent) 3-form, which means that the volume element is well defined, that is unique. We denote it by Θ and choose

$$\Theta = -\frac{i}{2\epsilon}\,\theta^2\theta^3\theta^2 \tag{2.15}$$

in order that Θ reduce to $\theta^1\theta^2\theta^3$ in the commutative limit: this is important for example, to properly identify the Lagrangian. To find the Lagrangian we need in addition the Hodge-$*$ operation. One possibility to define it, proposed in [33], is

$$*\,[\theta^1, \theta^2] = 2\theta^3, \qquad *[\theta^2, \theta^3] = 2\theta^1, \qquad *[\theta^3, \theta^1] = 2\theta^2. \tag{2.16}$$

We shall use this definition: a discussion of its properties is given in the Appendix.

On noncommutative spaces one can define other differential-geometric quantities like the affine connection, torsion and curvature. The connection 1-form used in [31]

to define the paralell transport on the truncated Heisenberg space is given by

$$\begin{aligned}
\omega_{12} &= -\omega_{21} = (-\frac{\mu}{2} + 2i\epsilon p_3)\theta^3 = \mu\,(\frac{1}{2} - 2\mu z)\theta^3 \\
\omega_{13} &= -\omega_{31} = \frac{\mu}{2}\,\theta^2 + 2i\epsilon p_2\theta^3 = \frac{\mu}{2}\,\theta^2 + 2\mu^2 x\,\theta^3 \\
\omega_{23} &= -\omega_{32} = -\frac{\mu}{2}\,\theta^1 - 2i\epsilon p_1\theta^3 = -\frac{\mu}{2}\,\theta^1 + 2\mu^2 y\,\theta^3.
\end{aligned} \tag{2.17}$$

It can be shown that this paralell transport preserves lengths, that is connection (2.17) is metric compatible. Having the connection, the torsion and the curvature tensors are defined as usual:

$$T^\alpha = d\theta^\alpha + \omega^\alpha{}_\beta\theta^\beta, \qquad \Omega^\alpha{}_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma\omega^\gamma{}_\beta. \tag{2.18}$$

If we denote

$$\Omega^\alpha{}_\beta = \frac{1}{2}\,R^\alpha{}_{\beta\gamma\delta}\theta^\gamma\theta^\delta \tag{2.19}$$

then by contractions we obtain the Ricci tensor and the scalar curvature. In our case,

$$R = \eta^{\beta\delta}R^\alpha{}_{\beta\alpha\delta} = \frac{15\mu^2}{2} - 4\mu^3 z - 8\mu^4(x^2+y^2). \tag{2.20}$$

Calculating the torsion 2-form we obtain

$$\begin{aligned}
T^1 &= -i\,\frac{\epsilon\mu}{2}\,(1-2\mu z)\,[\theta^1,\theta^3] \\
T^2 &= -i\,\frac{\epsilon\mu}{2}\,(1-2\mu z)\,[\theta^2,\theta^3] \\
T^3 &= -i\epsilon\mu^2 x\,[\theta^2,\theta^3] + i\epsilon\mu^2 y\,[\theta^1,\theta^3],
\end{aligned} \tag{2.21}$$

and its dual

$$\begin{aligned}
*T^1 &= i\epsilon\mu(1-2\mu z)\,\theta^2 \\
*T^2 &= -i\epsilon\mu(1-2\mu z)\,\theta^1 \\
*T^3 &= -2i\epsilon\mu^2 x\,\theta^1 - 2i\epsilon\mu^2 y\,\theta^2.
\end{aligned} \tag{2.22}$$

3 Spinors on the truncated Heisenberg space

Let us recall briefly the commutative action for the Dirac spinors in the external gravitational field, to fix the notation. We have the euclidean space,

$$\{\gamma^\alpha,\gamma^\beta\} = 2\delta^{\alpha\beta}, \tag{3.23}$$

$\alpha,\beta = 1,2,3,$ so the γ-matrices are hermitian. The Dirac spinor $\psi(x)$ transforms, under the local frame rotations, in the spinor represenatation: for an infinitesimal

rotation $\Lambda^\alpha{}_\beta = \delta^\alpha{}_\beta + \lambda^\alpha{}_\beta$ the representation is given by $S(\Lambda) = 1 + \frac{1}{4}\lambda_{\alpha\beta}\gamma^\alpha\gamma^\beta$. The covariant derivative is therefore

$$D\psi = d\psi + \frac{1}{4}\,\omega^\delta{}_\gamma\gamma_\delta\gamma^\gamma\psi = (D_\alpha\psi)\theta^\alpha, \tag{3.24}$$

that is,

$$D_\alpha\psi = e_\alpha\psi + \Gamma_\alpha\psi, \qquad D_\alpha\bar{\psi} = e_\alpha\psi - \bar{\psi}\Gamma_\alpha, \qquad \Gamma_\alpha = \frac{1}{4}\,\omega^\delta{}_{\alpha\beta}\gamma_\delta\gamma^\beta. \tag{3.25}$$

Since the group generators are hermitian, $\bar{\psi} = \psi^\dagger$. The Dirac operator, $\not{D} = \gamma^\alpha D_\alpha$ defines the action

$$S = \int \sqrt{g}\,\bar{\psi}\,(i\not{D} - m)\psi. \tag{3.26}$$

It can be seen easily by partial integration that (3.26) is real only if the torsion vanishes, more precisely if

$$\omega^\alpha{}_{\alpha\gamma} = \frac{1}{\sqrt{g}}\partial_\mu(e^\mu_\gamma\sqrt{g}). \tag{3.27}$$

If not one defines the spinor action by symmetrisation,

$$\mathcal{S} = \frac{1}{2}\,(S + S^*). \tag{3.28}$$

Action (3.26) can be rewritten in the language of forms, [36]. If we introduce a matrix-valued 1-form $V = \theta^\alpha\gamma_\alpha$, in d dimensions we have

$$\int \mathrm{Tr}\,(D\psi)\bar{\psi}\,VV\ldots V\gamma_5 = -i(d-1)!\int \Theta\,\bar{\psi}\gamma^\alpha(D_\alpha\psi), \tag{3.29}$$

where Tr is the trace in γ-matrices; the product $VV\ldots V$ contains $(d-1)$ factors and thus the volume d-form Θ appears under the integral. In commutative case all 1-forms anticommute so the hermitian part of (3.29) is

$$\mathcal{S}_{kin} = \frac{1}{2}\int \mathrm{Tr}\,\big((D\psi)\bar{\psi} - \psi(D\bar{\psi})\big)\,VV\ldots V\gamma_5. \tag{3.30}$$

Similarly, the mass term can be written as

$$m\int \mathrm{Tr}\,\psi\bar{\psi}VVV\ldots V\gamma_5 = -id!\int \Theta\,m\bar{\psi}\psi, \tag{3.31}$$

where now the product of 1-forms $VVV\ldots V$ has d factors.

Since we wish to construct a Dirac spinor on three-dimensional space and then reduce to two dimensions we need both representations. In two and three space-time dimensions the irreducible spinor representations are two-dimensional. In 2d a natural choice for the γ-matrices are Pauli matrices

$$\gamma_1 = \sigma_1, \quad \gamma_2 = \sigma_2. \tag{3.32}$$

From γ_1 and γ_2 we obtain the γ_5-matrix (which we denote by γ_3),

$$\gamma_3 = -i\gamma_1\gamma_2 = \sigma_3, \tag{3.33}$$

it is the chirality operator in two dimensions. The representation is, up to unitary equivalence, unique. In three dimensions the Pauli matrices $\gamma_\alpha = \sigma_\alpha$, $\alpha = 1,2,3$ also give a representation. For the γ_5 we have

$$\gamma_4 = -i\gamma_1\gamma_2\gamma_3 = 1. \tag{3.34}$$

The other, inequivalent representation is $\tilde{\gamma}_1 = \sigma_1$, $\tilde{\gamma}_2 = \sigma_2$, $\tilde{\gamma}_3 = -\sigma_3$, and yields $\tilde{\gamma}_4 = -1$. Thus on a three-dimensional space the spinor action is the sum of terms

$$\mathcal{S}_{kin} = \frac{1}{4}\int \mathrm{Tr}\,\Big((D\psi)\bar{\psi} - \psi(D\bar{\psi})\Big)VV, \qquad \mathcal{S}_{mass} = \frac{i}{6}\int \mathrm{Tr}\,\psi\bar{\psi}\,VVV. \tag{3.35}$$

Let us construct the Dirac action on the truncated Heisenberg space. For simplicity we first calculate

$$S^*_{kin} = -\frac{1}{2}\int \mathrm{Tr}\,\psi(D\bar{\psi})\,VV = -\frac{1}{2}\int \mathrm{Tr}\ \Xi_\alpha\gamma_\beta\gamma_\gamma\theta^\alpha\theta^\beta\theta^\gamma, \tag{3.36}$$

and then symmetrise; we introduce

$$\Xi_\alpha = \psi(D_\alpha\bar{\psi}) = \psi\,\big((e_\alpha\bar{\psi}) - \bar{\psi}\Gamma_\alpha\big). \tag{3.37}$$

Using

$$\begin{aligned}
&\Xi_1 = \psi\,\Big((e_1\bar{\psi}) + \frac{i\mu}{4}\,\bar{\psi}\gamma_1\Big), \qquad \Xi_2 = \psi\,\Big((e_2\bar{\psi}) + \frac{i\mu}{4}\,\bar{\psi}\gamma_2\Big), \\
&\Xi_3 = \psi\,\Big((e_3\bar{\psi}) - \frac{i\mu}{4}\,\bar{\psi}\gamma_3 + i\mu^2\bar{\psi}(x\gamma_2 - y\gamma_1 + z\gamma_3)\Big)
\end{aligned} \tag{3.38}$$

and the algebra of 1-forms we obtain

$$\begin{aligned}
S^*_{kin} &= \frac{1}{2}\int \Theta\,\mathrm{Tr}\,\big(i\,\Xi_1\gamma_1 + i\,\Xi_2\gamma_2 + i(1-\epsilon^2)\,\Xi_3\gamma_3 - \epsilon\,\Xi_1\gamma_2 + \epsilon\,\Xi_2\gamma_1\big) \\
&= \frac{1}{2}\int \Theta\,\Big(i(e_1\bar{\psi})\gamma_1\psi + i(e_2\bar{\psi})\gamma_2\psi + i(1-\epsilon^2)(e_3\bar{\psi})\gamma_3\psi \\
&\quad - \frac{\mu}{4}\,(1+\epsilon^2)\bar{\psi}\psi + \frac{\mu\epsilon}{2}\,\bar{\psi}\gamma_3\psi - \mu^2(1-\epsilon^2)\bar{\psi}z\psi \\
&\quad - \epsilon(e_1\bar{\psi})\gamma_2\psi + \epsilon(e_2\bar{\psi})\gamma_1\psi - i\mu^2(1-\epsilon^2)\bar{\psi}(x\gamma_1 + y\gamma_2)\psi\Big).
\end{aligned} \tag{3.39}$$

Terms in the last line are imaginary. Therefore taking the hermitian part, when we reduce (3.39) to subspace $z = 0$, $e_3\bar{\psi} = 0$, we obtain

$$\mathcal{S}|_{kin} = \frac{1}{2}\int \Theta\,\Big(i\bar{\psi}\gamma^\alpha(e_\alpha\psi) - i(e_\alpha\bar{\psi})\gamma^\alpha\psi + \frac{1}{2}\,\mu(1+\epsilon^2)\bar{\psi}\psi - \mu\epsilon\,\bar{\psi}\gamma_3\psi\Big) \tag{3.40}$$

from the kinetic part of the 3d action; summation is now in $\alpha = 1, 2$. We can observe that, as in the scalar case, a part of the connection terms after dimensional reduction manifest as mass. In a similar way from the 3d mass term (3.31) we have

$$\mathcal{S}|_{mass} = i\,\frac{m}{6}\int \mathrm{Tr}\,\psi\bar{\psi}\,VVV = -m\int \Theta\,\Big((1-\frac{\epsilon^2}{3})\bar{\psi}\psi - \frac{2\epsilon}{3}\,\bar{\psi}\gamma_3\psi\Big); \qquad (3.41)$$

it looks the same before and after dimensional reduction. In fact being more precise, the volume element before dimensional reduction is $\Theta^{(3)}$, and after it is $\Theta^{(2)}$. This is not explicitly stressed in our notation; it is also understood implicitly that the integral over the third direction renormalizes the field ψ and, in the interacting case, the coupling constants.

In the absence of noncommutativity, $\epsilon = 0$, the mass term reduces to the usual one. But noncommutative algebra (2.6) is not invariant under the space inversion: this is reflected in the $\bar{\psi}\gamma_3\psi$ terms in (3.40) and (3.41). Therefore the spinors of different chirality have different masses in (3.41): for $\epsilon = 1$ for example ψ_R is massless while $m_L = 4m/3$. The kinetic term also generates different masses for spinors of opposite chirality, $m_{L,R} = \mu(1\pm\epsilon)^2/4$.

But as stressed before, the minimal coupling of spinors to the curved background does not give coordinate-dependent terms: their contributions are imaginary. Therefore, if we wish to introduce the 'square root of the harmonic potential', we have to include nonminimal interaction with the torsion. Various interaction terms are possible, [37]; from dimensional analysis and invariance arguments it follows that they are linear in torsion and bilinear in spinors. Interaction terms in 3d are

$$S'_{tor} = \int \mathrm{Tr}\,\psi\,\bar{\psi}\,T_\alpha\gamma^\alpha V, \qquad S''_{tor} = \int \mathrm{Tr}\,\psi\,\bar{\psi}\,(*T_\alpha)\gamma^\alpha VV, \qquad (3.42)$$

but it turns out that they are proportional,

$$S''_{tor} = 2iS'_{tor}. \qquad (3.43)$$

The calculation gives

$$\begin{aligned} S'_{tor} = {} & 2\epsilon\int \Theta\,\bar{\psi}\Big((\epsilon-\gamma_3)(\mu-2\mu^2 z) + (\mu^2 x\gamma_2 - \mu^2 y\gamma_1)\Big)\psi \\ & - 2i\epsilon^2\int \Theta\,\bar{\psi}\Big(\mu^2 x\gamma_1 + \mu^2 y\gamma_2\Big)\psi, \end{aligned} \qquad (3.44)$$

so we have two independent terms, the real and the imaginary part of S'_{tor}. After the reduction to $z = 0$ we obtain

$$\mathcal{S}|_{tor} = \frac{a\mu}{2}\int \Theta\,(\epsilon\,\bar{\psi}\psi - \bar{\psi}\gamma_3\psi) + \frac{1}{2}\int \Theta\,\bar{\psi}(a\epsilon_{\alpha\beta} + b\delta_{\alpha\beta})\mu^2 x^\alpha\gamma^\beta\psi, \qquad (3.45)$$

where a and b are arbitrary real coefficients and the summation is, as in (3.40), in $\alpha = 1, 2$.

In conclusion, the general Lagrangian which describes the 3d Dirac spinors, after the reduction to two dimensions, is given by

$$\begin{aligned}\mathcal{L} &= \mathcal{L}|_{kin} + \mathcal{L}|_{mass} + \mathcal{L}|_{tor} = \\ &= \frac{1}{2}\Big(i\bar{\psi}\gamma^\alpha(e_\alpha\psi) - i(e_\alpha\bar{\psi})\gamma^\alpha\psi\Big) + \frac{1}{2}\,\bar{\psi}(a\epsilon_{\alpha\beta} + b\delta_{\alpha\beta})\,\mu^2 x^\alpha\gamma^\beta\psi \\ &\quad - m\Big((1-\frac{\epsilon^2}{3})\bar{\psi}\psi - \frac{2\epsilon}{3}\,\bar{\psi}\gamma_3\psi\Big) + \frac{\mu}{4}\,\Big((1+2a\epsilon+\epsilon^2)\bar{\psi}\psi - 2(\epsilon+a)\,\bar{\psi}\gamma_3\psi\Big).\end{aligned} \tag{3.46}$$

Writing (3.46) in the form $\mathcal{L} = \bar{\psi}\not{D}\psi$ we find the corresponding Dirac operator

$$\not{D} = i\gamma^\alpha e_\alpha - A - B\gamma_3 + \frac{1}{2}(a\epsilon_{\alpha\beta} + b\delta_{\alpha\beta})\,\mu^2 x^\alpha\gamma^\beta, \tag{3.47}$$

with

$$A = \frac{1}{3}\,(3-\epsilon^2)m - \frac{1}{4}\,(1+2a\epsilon+\epsilon^2)\mu, \quad B = -\frac{2\epsilon}{3}\,m + \frac{1}{2}\,(a+\epsilon)\mu. \tag{3.48}$$

The square of this operator is

$$\begin{aligned}\not{D}^2 &= -e_\alpha e^\alpha - 2A\gamma^\alpha i e_\alpha + \frac{1}{4}\,(a^2+b^2)\mu^4 x_\alpha x^\alpha \\ &\quad + (A^2+B^2) + \big(2AB - \mu^2 a - \frac{1}{4}\,\mu^2\epsilon(a^2+b^2)\big)\gamma_3 \\ &\quad - A(a\epsilon^{\alpha\beta} + b\delta^{\alpha\beta})\mu^2 x_\alpha\gamma_\beta + \frac{1}{2}\,(a\epsilon^{\alpha\beta} + b\delta^{\alpha\beta})\{ie_\alpha, \mu^2 x_\beta\}.\end{aligned} \tag{3.49}$$

The obtained $\not{D}^2$ is a generalisation of the usual Lichnerowicz spinor laplacian: it contains an additional dependence on the connection coming from the interaction with the torsion. In addition, there are terms induced by dimensional reduction.

4 Concluding remarks

The main objective of calculations presented in this paper was to extend our previous work and build a consistent geometrical action for the Dirac spinors on the truncated Heisenberg space, and to reduce it in the next step to the Heisenberg subspace. As we wished to relate this theory eventually to the Grosse-Wulkenhaar action, we wanted to include coordinates explicitly: therefore we introduced the nonminimal interaction of spinors with the torsion. The result, somewhat unexpected, is indeed nice: the action (3.46) which we found is in fact equivalent to the noncommutative extension of the Gross-Neveu action (1.3) proposed by Vignes-Tourneret, which is renormalisable. Renormalisability of the noncommutative GN model does not trivially reduce to renormalisability of the GW model since one is not a simple square of the other, [14].

To effect the mentioned equivalence we need in fact only one of the interaction terms: we set $b = 0$. Comparing notations we identify noncommutativity θ of [15]

as $\theta = -\bar{k}$. Then the remaining parameters of the fermion actions (1.3) and (3.46) are associated as

$$\tilde{m} = A, \qquad \kappa = B, \qquad \Omega = \frac{a\epsilon}{4}, \tag{4.50}$$

where A and B are given in (3.48).

It is further interesting to notice that the harmonic term in the Grosse-Wulkenhaar model can be obtained, alternatively, as an interaction of the scalar field with the torsion. Therefore 'geometrisation' of the mentioned renormalisable actions can be formulated in terms of the interaction with torsion solely. Indeed, from (2.21-2.22) on the truncated Heisenberg space we find

$$(*T_\alpha)\, T^\alpha = T^\alpha\, (*T_\alpha) = -2\mu^2\epsilon^2\Theta\Big((1-2\mu z)^2 + 2\mu^2(x^2+y^2) - 2\epsilon^2(1-\mu z)\Big) \tag{4.51}$$

so the corresponding interaction Lagrangian with the scalar field ϕ, reduced to 2d, is

$$\mathcal{L}_{\phi,tor} = -\xi\mu^2\epsilon^2\Big(2(\mu^2x^2+\mu^2y^2) + 1 - 2\epsilon^2\Big)\phi^2, \tag{4.52}$$

where $\xi/2$ is the coupling constant. As in the case of the coupling with curvature [31], the interaction with torsion introduces the harmonic potential and modifies the mass. The present result seemingly suggests that the torsion in a way has a primary role in the analysed set of models. Unfortunately it cannot couple to nonabelian gauge fields in a gauge invariant way, and on a noncommutative space all gauge fields including the $U(1)$ are nonabelian. The question whether the torsion might improve properties of the gauge models needs perhaps some further clarification, maybe in the view of the dimensional reduction procedure.

There are other effects contained in our result which deserve further investigation: the creation of mass and the parity breaking. The fact that the gravitational field (seen as curvature, or torsion) manifests itself as inertia, that is mass, is intuitively clear. As an additional possible source of the particle mass we have here the dimensional reduction, but as the extra dimension is not compact, its volume just renormalises the wave function and the couplings. The parity breaking is also not hard to understand. Since we start from a 3d space which is not invariant under the space inversion, the spinor Lagrangian does not have the symmetry either and the property remains after reduction to 2d. It is manifested as a difference between masses of the right and the left components of the spinor field: we have

$$m_{R,L} = A \pm B = \frac{m}{3}\,(1\mp\epsilon)(3\pm\epsilon) - \frac{\mu}{4}\,(1\mp\epsilon)(1\mp\epsilon\mp 2a). \tag{4.53}$$

Interestingly, the effect of the parity breaking can be produced solely by coupling to the torsion, which can be seen in the previous formula by putting $\epsilon = 0$, $a \neq 0$. Both of the mentioned effects give interesting possibilities for the model building in particle physics: they might provide us with new variants of the see-saw mechanism as well.

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Appendix: Hodge-$*$ and the volume form

A part of the algebra of 3-forms (2.14), rewritten as

$$\begin{aligned} &[\theta^1,\theta^2]\,\theta^3 = \theta^3[\theta^1,\theta^2] = 2(1-\epsilon^2)\Theta, \\ &[\theta^2,\theta^3]\,\theta^1 = \theta^1[\theta^2,\theta^3] = 2\Theta, \qquad\qquad [\theta^3,\theta^1]\,\theta^2 = \theta^2[\theta^3,\theta^1] = 2\Theta, \end{aligned} \tag{4.54}$$

suggests definition (2.16) the $*$-operation. This definition is further in accordance with the usual convention for the double action of the Hodge-$*$, which in three euclidean dimensions is $*(*\omega) = \omega$ for all p-forms ω. It looks however as if (2.16) changes the usual rules related to the volume element as for example

$$\left(*\tfrac{1}{2}\,[\theta^1,\theta^2]\right)\,\tfrac{1}{2}\,[\theta^1,\theta^2] = (1-\epsilon^2)\,\Theta, \qquad \left(*\tfrac{1}{2}\,[\theta^2,\theta^3]\right)\,\tfrac{1}{2}\,[\theta^2,\theta^3] = \Theta. \tag{4.55}$$

At this point we should recall that, in noncommutative case, the commutators of 1-forms are not natural as a basis in the space of 2-forms: we should rather use the twisted commutators, $\tilde\theta^{\alpha\beta} \equiv P^{\alpha\beta}{}_{\gamma\delta}\theta^\gamma\theta^\delta$, as twisted commutators enclose the properties of the noncommutative product. We have, [33]

$$\begin{aligned} \theta^1\theta^2 = \tilde\theta^{12} = P^{12}{}_{\gamma\delta}\theta^\gamma\theta^\delta &= \frac{1}{2}\,[\theta^1,\theta^2] \\ \theta^1\theta^3 = \tilde\theta^{13} = P^{13}{}_{\gamma\delta}\theta^\gamma\theta^\delta &= \frac{1}{2}\,[\theta^1,\theta^3] + \frac{i\epsilon}{2}\,[\theta^2,\theta^3] \\ \theta^2\theta^3 = \tilde\theta^{23} = P^{23}{}_{\gamma\delta}\theta^\gamma\theta^\delta &= \frac{1}{2}\,[\theta^2,\theta^3] - \frac{i\epsilon}{2}\,[\theta^1,\theta^3]. \end{aligned} \tag{4.56}$$

The main drawback of the basis $\{\tilde\theta^{\alpha\beta}\}$ it that is not hermitian. Applied to this basis elements the Hodge-$*$ gives

$$(*\tilde\theta^{12})\,\theta^{12} = (*\tilde\theta^{13})\,\theta^{13} = (*\tilde\theta^{23})\,\theta^{23} = \theta^1\theta^2\theta^3 \tag{4.57}$$

as one would expect; also, the order of factors does not matter. From (4.57) we see that in fact the volume 3-form should have been identified as

$$\widetilde\Theta = \theta^1\theta^2\theta^3 = (1-\epsilon^2)\Theta. \tag{4.58}$$

However because of non-hermiticity, we have for example $(*\tilde\theta^{13})\,\tilde\theta^{12} \neq 0$ but rather

$$(*\tilde\theta^{13})\,\tilde\theta^{12} + \tilde\theta^{12}\,(*\tilde\theta^{13}) = 0. \tag{4.59}$$

Analogous relations hold for other components. We have not in our calculation redefined the volume 3-form Θ to $\widetilde\Theta$, as this redefinition changes only the overall factor in the action. But formula (4.58) shows that the limit to the matrix case, $\epsilon = 1$, is not smooth: the space of p-forms becames a kind of fragmented. In a similar way the commutative limit, $\epsilon = 0$, is singular because in this limit the momenta p_α, (1.5) diverge. Perhaps a more detailed analysis of the tangent and cotangent spaces for $\epsilon = 1$ could reveal some interesting or characteristic properties of the matrix geometries or of the quantum groups, of which the truncated Heisenbeg algebra is one, somewhat exotic, example.

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